Type inference in context

Adam Gundry
University of Strathclyde
Microsoft Research PhD Scholar

Conor McBride
University of Strathclyde

James McKinna
Radboud University Nijmegen

MSFP

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Two kinds of problems

- Hindley-Milner type inference
  - λ-calculus with let-definitions
  - Parametric polymorphism

$$\text{let } f \coloneqq \lambda\, x \,.\, x \text{ in } f\ f$$

- First-order unification
  - Solves equations between types

$$\alpha \to \alpha \equiv (\beta \to \beta) \to \gamma$$

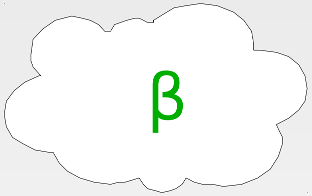
Traditional contexts

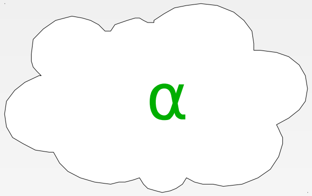
- Contexts explain term variables

$$\Gamma, f : \alpha \to \alpha, t : \beta \to \beta$$

Traditional contexts

- Contexts explain term variables
- Type variables float in space

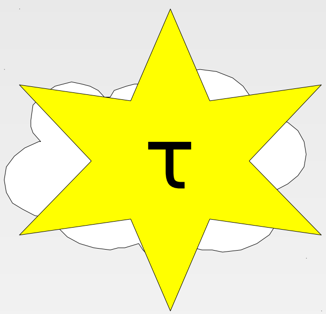


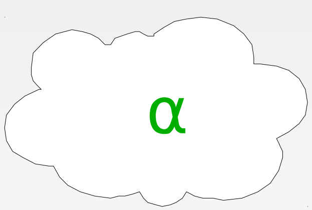


$$\Gamma, f : \alpha \to \alpha, t : \beta \to \beta$$

Traditional contexts

- Contexts explain term variables
- Type variables float in space
- They are given meaning by **substitution**





$$\Gamma, f : \alpha \rightarrow \alpha, t : \tau \rightarrow \tau$$

Type variables in the context

- We add type variables to the context

$$\Gamma, \quad f : \alpha \to \alpha, \quad t : \beta \to \beta$$

Type variables in the context

- We add type variables to the context

$$\Gamma, \alpha := ?, f : \alpha \to \alpha, \beta := ?, t : \beta \to \beta$$

Type variables in the context

- We add type variables to the context
- Give them meaning by **definition**

$$\Gamma, \alpha := ?, f : \alpha \to \alpha, \beta := \tau, t : \beta \to \beta$$

Type variables in the context

- We add type variables to the context
- Give them meaning by **definition**
- Work in the induced equational theory
- Explicitly scoped substitution in triangular form

$$\Gamma, \alpha \coloneqq ?, f : \alpha \to \alpha, \beta \coloneqq \tau, t : \beta \to \beta$$

Declarations and judgments

Declaration	Judgment
$\alpha \coloneqq ?$	$\Gamma \vdash \alpha \equiv \alpha$
$\alpha \coloneqq \tau$	$\Gamma \vdash \alpha \equiv \tau$
$f : \alpha \to \alpha$	$\Gamma \vdash f : \alpha \to \alpha$

Example of unification

To infer the type of the application

$$f\ \ t$$

we have to solve the unification problem

$$\alpha \to \alpha \equiv (\beta \to \beta) \to \gamma$$

where γ is a fresh type variable

Unification algorithm

unify :: Type → Type → StateT Context Maybe ()

Unification algorithm

unify :: Type → Type → Context → Maybe Context

Unification in the context

$$\Gamma, \quad \alpha \coloneqq ?, \quad f, \quad \beta \coloneqq ?, \quad t, \quad \gamma \coloneqq ?$$

$$\alpha \to \alpha \equiv (\beta \to \beta) \to \gamma$$

Unification in the context

$$\Gamma, \quad \alpha := ?, \quad f, \quad \beta := ?, \quad t, \quad \gamma := ?$$

$$\alpha \equiv \beta \to \beta$$

$$\alpha \equiv \gamma$$

Unification in the context

$$\Gamma, \quad \alpha := ?, \quad f, \quad \beta := ?, \quad t, \quad \gamma := ?$$

$\alpha \equiv \beta \to \beta$

$\alpha \equiv \gamma$

Unification in the context

$\Gamma, \quad \alpha := ?, \quad f, \quad \beta := ?, \quad t, \quad \gamma := ?$

$\alpha \equiv \beta \to \beta$

$\alpha \equiv \gamma$

Unification in the context

$$\Gamma, \quad \alpha := ?, \quad f, \quad \beta := ?, \quad t, \quad \gamma := ?$$

$$\alpha \equiv \beta \rightarrow \beta$$

$$\alpha \equiv \gamma$$

Unification in the context

$\Gamma, \quad \alpha := ?, \quad f, \quad t, \quad \gamma := ?$

$\beta := ?$ | $\alpha \equiv \beta \to \beta$

$\alpha \equiv \gamma$

Unification in the context

$$\Gamma, \quad \alpha := ?, \quad f, \quad t, \quad \gamma := ?$$

$\beta := ?$ | $\alpha \equiv \beta \rightarrow \beta$

$\alpha \equiv \gamma$

Unification in the context

$$\Gamma, \quad \beta \coloneqq ?, \quad \alpha \coloneqq ?, \quad f, \quad t, \quad \gamma \coloneqq ?$$

$$\alpha \equiv \beta \to \beta$$

$$\alpha \equiv \gamma$$

Unification in the context

$$\Gamma, \quad \beta := ?, \quad \alpha := \beta \to \beta, \quad f, \quad t, \quad \gamma := ?$$

$$\alpha \equiv \gamma$$

Unification in the context

$$\Gamma, \quad \beta := ?, \quad \alpha := \beta \to \beta, \quad f, \quad t, \quad \gamma := ?$$

$$\alpha \equiv \gamma$$

Unification in the context

$$\Gamma, \quad \beta := ?, \quad \alpha := \beta \to \beta, \quad f, \quad t, \quad \gamma := \alpha$$

Solution strategy

- Refine context in small steps to solve problem
- Easy to verify that each step is sound
- Minimal commitment at each step
- Why does this give most general solutions?

Information increase

- An **information increase** $\theta : \Gamma \sqsubseteq \Delta$ is
  - a substitution from type variables of Γ to types of Δ
  - such that every declaration in Γ
    holds as a judgment in Δ (under the substitution)
- If $\alpha \coloneqq \tau \in \Gamma$ then $\Delta \vdash \theta(\alpha \equiv \tau)$
- We only use the identity substitution,
  but are general with respect to any substitution

Examples of information increase

- Adding fresh variables

$$\Gamma \sqsubseteq \Gamma, \alpha \coloneqq ?, \beta \coloneqq ?$$

- Defining a previously undefined type variable

$$\Gamma, \alpha \coloneqq ? \sqsubseteq \Gamma, \alpha \coloneqq \tau$$

- Substituting out a definition

$$[\tau/\alpha] : \Gamma, \alpha \coloneqq \tau, \beta \coloneqq \alpha \sqsubseteq \Gamma, \beta \coloneqq \tau$$

Stability

- A judgment is **stable** if it is preserved by information increase
- “Once solved, always solved”
- Stability by construction: context access just looks up facts about variables
- Minimal-commitment solutions to stable problems are most general

Type inference

infer :: Term → StateT Context Maybe Type

Type inference

infer :: Term → Context → Maybe (Type, Context)

Type inference: term variables

$$x :: \forall\, \alpha\, \beta\, .\, \alpha \to \beta \to \alpha \quad \in \quad \Gamma$$

- Contexts give type-schemes to term variables

Type inference: term variables

$$\Gamma \qquad\qquad \vdash \; x :: \forall\, \alpha\, \beta\, .\, \alpha \to \beta \to \alpha$$

- Instantiate type-scheme with fresh variables

Type inference: term variables

$$\Gamma, \alpha' \coloneqq ? \qquad \vdash \; x :: \forall \beta \, . \, \alpha' \to \beta \to \alpha'$$

- Instantiate type-scheme with fresh variables

Type inference: term variables

$$\Gamma, \alpha' := ?, \beta' := ? \vdash x : \alpha' \to \beta' \to \alpha'$$

- Instantiate type-scheme with fresh variables

Generalisation

- How to generalise types in let-expressions?
- Traditionally, compare sets of free variables

$$\frac{A \vdash e' : \tau' \qquad A_x \cup \{x : \sigma\} \vdash e : \tau}{A \vdash \text{let } x := e' \text{ in } e : \tau} \; \sigma = gen(A, \tau')$$

$$gen(A, \tau) = \begin{cases} \forall \vec{\alpha_i}.\tau & (FV(\tau) \setminus FV(A) = \{\alpha_1, \cdots, \alpha_n\}) \\ \tau & (FV(\tau) \setminus FV(A) = \emptyset) \end{cases}$$

Generalisation

- Structure on type variables makes it easy
- 'Skim off' type variables from the local, unconstrained end of the context
- Use a marker to record where to stop
- Unification may move variables past the marker

Type inference: let expressions

- To infer the type of let expressions:

$$\text{let } y \coloneqq t \text{ in } e$$

- Place a marker in the context
- Infer the type of the definition t
- Generalise over type variables
- Extend the context with a type-scheme for y
- Infer the type of the body e

Type inference: let expressions

$\Gamma \vdash \text{let } f \coloneqq \lambda x . x \text{ in } f\ f : ?$

Type inference: let expressions

$$\Gamma \,; \qquad \vdash \text{let } f \coloneqq \lambda x\,.\,x \text{ in } f\ f : ?$$

- Place a marker ; in the context
- This records where to stop generalising

Type inference: let expressions

$$\Gamma\,; \qquad\qquad \vdash \lambda x . x : ?$$

- Infer the type of the let-definition

Type inference: let expressions

$$\Gamma \; ; \; \alpha \coloneqq \; ? \qquad \vdash \; \lambda \, x \, . \, x : \alpha \to \alpha$$

- Infer the type of the let-definition

Type inference: let expressions

$$\Gamma\ ;\ \alpha \coloneqq ? \qquad \vdash\ \lambda\, x\, .\, x : \alpha \to \alpha$$

- Generalise back to the marker

Type inference: let expressions

$\Gamma \; ; \qquad \vdash \lambda x . x :: \forall \alpha . \alpha \to \alpha$

- Generalise back to the marker

Type inference: let expressions

$$\Gamma, f :: \forall \alpha . \alpha \to \alpha \vdash f\ f : ?$$

- Assign type-scheme to the let-bound variable
- Infer the type of the let-body

Giving up some freedom

- Typing for let expressions is non-compositional
- We must restrict the information increase relation so terms have principal types
- Forbid assignment of more general types to variables in the context

$$\Gamma, x : \alpha \not\sqsubseteq \Gamma, x :: \forall \alpha . \alpha$$

What have we achieved?

- A methodology for problem solving in contexts
- Connected soundness and generality of algorithms for problem solving
- More intuitive account of generalisation

How to solve problems in contexts

- Define properties of variables
- Define judgments: stable by construction
- Show solution steps are information increases
- Restrict freedom so unique solutions exist
- Minimal commitment yields principal solutions

Future directions

- Formal correctness proof?
- Investigate more complex type systems
- Undecidable constraint systems
- Represent syntactic context explicitly (zipper)

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Free monad on derivations

- Stability extends substitution on types to substitution on typing derivations
- Derivations have a free monad structure:
  - Return takes a variable to the derivation that looks up the variable in the context
  - Bind is substitution of sub-derivations